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Applied Mathematics Letters 19 (2006) 472–477
Applied Mathematics Letters
www.elsevier.com/locate/aml

On graphs with equal total domination and connected domination numbers

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Received 30 June 2005; accepted 30 June 2005

Abstract

A subset S of V is called a *total dominating set* if every vertex in V is adjacent to some vertex in S. The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality taken over all total dominating sets of G. A dominating set is called a *connected dominating set* if the induced subgraph $\langle S \rangle$ is connected. The *connected domination number* $\gamma_c(G)$ of G is the minimum cardinality taken over all minimal connected dominating sets of G. In this work, we characterize trees and unicyclic graphs with equal total domination and connected domination numbers.

Keywords: Tree; Unicyclic graph; Total domination number; Connected domination number

1. Introduction

By a graph we mean a finite, undirected graph without loops or multiple edges. Terms not defined here are used in the sense of Arumuram [1] and Harary [2].

Let $G = (V, E)$ be a simple graph of order n. The degree, neighborhood and closed neighborhood of a vertex v in the graph G are denoted by $d(v)$, $N(v)$ and $N[v] = N(v) \cup \{v\}$, respectively. For a subset S of V, $N(S)$ denotes the set of all vertices adjacent to some vertex in S and $N[S] = N(S) \cup S$. The graph induced by $S \subseteq V$ is denoted by $\langle S \rangle$. The minimum degree and maximum degree of the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Let P_n and $K_{1,n-1}$ denote the path and star with n vertices, respectively.

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doi:10.1016/j.aml.2005.06.016

A subset S of V is called a *dominating set* if every vertex in $V - S$ is adjacent to some vertex in S. The *domination number* $\gamma(G)$ of G is the minimum cardinality taken over all dominating sets of G. A subset S of V is called a *total dominating set* if every vertex in V is adjacent to some vertex in S. The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality taken over all total dominating sets of G. A dominating set is called a *connected dominating set* if the induced subgraph $\langle S \rangle$ is connected. The *connected domination number* $\gamma_c(G)$ of G is the minimum cardinality taken over all minimal connected dominating sets of G. A connected dominating set S with cardinality $\gamma_c(G)$ is called a γ_c-set. Let $S \subset V(G)$ and $x \in S$, we say that x has a *private neighbour* (with respect to S) if there is a vertex in $V(G) - S$ whose only neighbour in S is x. Let $PN(x, S)$ denote the private neighbours set of x with respect to S.

A vertex of degree one is called a *pendant vertex*. A vertex v of G is called a *support* if it is adjacent to a pendant vertex. Any vertex of degree greater than one is called an *internal vertex*. Let $L(G)$, $S(G)$ and $I(G)$ denote the sets of pendant vertices, support vertices and internal vertices of graph G, respectively. Let $C(G) = \{v \mid v \in I(G) - S(G)$, and there exists at least a component G_i of $G - \{v\}$ such that $|V(G_i) \cap I(G)| = 1\}$.

For any connected graph G a vertex $v \in V$ is called a *cutvertex* of G if $G - v$ is no longer connected. A graph G is called *unicyclic graph* if G contains exactly one cycle. Arumuram and Paulraj Joseph [1] have characterized trees and unicyclic graphs with equal domination and connected domination numbers.

Lemma 1 ([1]). *For a tree T of order $n \geq 3$, $\gamma_c(T) = \gamma(T)$ if and only if every internal vertex of T is a support.*

Lemma 2 ([1]). *Let G be a unicyclic graph with cycle C of length at least 5, and let X be the set of all vertices of degree 2 in C. Then $\gamma(G) = \gamma_c(G)$ if and only if the following conditions hold:*

(a) *Every vertex of degree at least 2 in $V - N[X]$ is a support.*
(b) *$\langle X \rangle$ is connected and $|X| \leq 3$.*
(c) *If $\langle X \rangle = P_1$ or P_3, both vertices in $N(X)$ of degree greater than 2 are supports and if $\langle X \rangle = P_2$, at least one vertex in $N(X)$ of degree greater than 2 is a support.*

Lemma 3 ([1]). *Let G be a unicyclic graph of order $n \geq 4$ with cycle C of length 3, and let X be the set of all vertices of degree 2 in C. Then $\gamma(G) = \gamma_c(G)$ if and only if the following conditions hold:*

(a) *Every vertex of degree at least 2 in $V - N[X]$ is a support.*
(b) *C contains exactly one vertex of degree at least 3 or every vertex of degree at least 3 in C is a support.*

Lemma 4 ([1]). *Let G be a unicyclic graph of order $n \geq 5$ with cycle C of length 4, and let X be the set of all vertices of degree 2 in C. Then $\gamma(G) = \gamma_c(G)$ if and only if the following conditions hold:*

(a) *Every vertex of degree at least 2 in $V - N(X)$ is a support.*
(b) *If $|X| = 1$, all the three remaining vertices of C are supports and if $|X| \geq 2$, C contains at least one support.*

In this work, we characterize trees and unicyclic graphs with equal total domination and connected domination numbers.

2. Main results

Suppose that T is a tree. If $|I(T)| \leq 1$, then T is a star and it is obvious that $\gamma_c(T) \neq \gamma_t(T)$.

Theorem 1. *Let T be a tree with $|I(T)| \geq 2$. Then $\gamma_c(T) = \gamma_t(T)$ if and only if $I(T) = S(T) \cup C(T)$.*

Proof. Since T is a tree, it follows that $I(T)$ is the unique minimum connected dominating set of G.

Now, let $\gamma_c(G) = \gamma_t(G)$. If $I(T) = S(T)$, then $C(T) = \emptyset$. It is obvious that $I(T) = S(T) \cup C(T)$. Without loss of generality, we can assume that $I(T) - S(T) \neq \emptyset$. For any $v \in I(T) - S(T)$, let $T_1, T_2, \ldots, T_{d(v)}$ denote the components of $T - \{v\}$. If every component T_i of $T - \{v\}$ satisfies $|V(T_i) \cap I(T)| \geq 2$, then $I(T) - \{v\}$ is a total dominating set of G with cardinality $\gamma_t(G) - 1$, which is a contradiction. Hence, $I(T) = S(T) \cup C(T)$.

Conversely, let S be a γ_t-set of G with minimum number of pendant vertices. Since $|I(T)| \geq 2$, it follows that $S \cap L(T) = \emptyset$ and $S(T) \subseteq S$. Since for any $v \in C(T)$ there exists at least a component T_i of $T - \{v\}$ such that $|V(T_i) \cap I(T)| = 1$, it follows that $v \in S$. Otherwise S is not a total dominating set of T, which is a contradiction. So, $C(T) \subseteq S$. Since $I(T) = S(T) \cup C(T)$, it follows that $I(T) \subseteq S$. That is $\gamma_t(T) \geq \gamma_c(T)$. Since $\gamma_t(G) \leq \gamma_c(G)$, it follows that $\gamma_c(G) = \gamma_t(G)$.

Let G be a unicyclic graph with cycle C_m, and let X be the set of all vertices of degree 2 in C_m. Without loss of generality, we can assume that $v_1, v_2, \ldots, v_t$ is the longest path in $\langle X \rangle$. Let $C_m = v_1, v_2, \ldots, v_t, v_{t+1}, \ldots, v_m, v_1$. If $\Delta(G) = n - 1$, then $\gamma_c(G) = 1$ and $\gamma_t(G) \neq \gamma_c(G)$.

Lemma 5. *Let C_m be a cycle with m vertices. Then $\gamma_t(G) = \gamma_c(G)$ if and only if $m = 4, 5, 6$.*

Lemma 6. *Let G be a unicyclic graph with cycle C_m. If $\Delta(G) \leq n - 2$ and $|X| = m - 1$, then $\gamma_t(G) = \gamma_c(G)$ if and only if the following conditions hold:*

(a) $3 \leq m \leq 6$.
(b) *Suppose $d(v_m) \geq 3$. Let $G' = G - \{v_1\}$. Then $I(G') = S(G') \cup C(G')$.*

Proof. Let $\gamma_t(G) = \gamma_c(G)$. By Lemma 5, it follows that $3 \leq m \leq 6$. It is obvious that $\gamma_c(G') = \gamma_c(G) = |I(G')|$. If $m = 6$, then $N(v_6) - \{v_1, v_5\} = L(G)$. Otherwise, $I(G') - \{v_5\}$ is a total dominating set of G with cardinality less than $\gamma_c(G)$, which is a contradiction. Hence, if $m = 6$, then (b) holds. If there exists a vertex $v \in I(G') - S(G') \cup C(G')$, then by Theorem 1 $\gamma_t(G') < \gamma_c(G')$. Since $\Delta(G) \leq n - 2$, it follows that $|I(G')| \geq 2$. Let S be a γ_t-set of G' such that $S \cap L(G') = \emptyset$. So, $S \subseteq I(G')$. If $v_m \in S$, then S is a total dominating set of G. Hence, $\gamma_t(G) \leq \gamma_t(G') < \gamma_c(G') = \gamma_c(G)$, which is a contradiction. If $v_m \notin S$, then $m = 5$ and v_m is not a support. Then $v_3, v_4 \in S$. If $N(v_5) \cap (S - \{v_4\}) \neq \emptyset$, then $(S - \{v_4\}) \cup \{v_2\}$ is a total dominating set of G with cardinality less than $\gamma_c(G)$, which is a contradiction. If $N(v_5) \cap (S - \{v_4\}) = \emptyset$, then $|S| \leq \gamma_c(G') - 2$ and $S \cup \{v_2\}$ is a total dominating set of G. Hence, $\gamma_t(G) \leq \gamma_t(G') + 1 < \gamma_c(G') = \gamma_c(G)$, which is a contradiction. So, $I(G') = S(G') \cup C(G')$.

Conversely, by Theorem 1, it follows that $\gamma_t(G') = \gamma_c(G')$. Since $\gamma_c(G') = \gamma_c(G)$, it follows that $\gamma_t(G') = \gamma_c(G)$. If $m = 3$, then it is obvious that $\gamma_t(G) = \gamma_t(G')$. That is $\gamma_t(G) = \gamma_c(G)$. If $m = 6$, then $N(v_6) - \{v_1, v_5\} = L(G)$ and $\gamma_t(G) = \gamma_c(G)$. If $m = 4, 5$, then let S be a γ_t-set of G such that $S \cap L(G) = \emptyset$. For any $v \in I(G) - V(C_m)$, since $v \in S(G') \cup C(G')$, it follows that $v \in S(G) \cup C(G)$ and $v \in S$. Since $|S \cap V(C_m)| \geq m - 2$, it follows that $\gamma_t(G) \geq \gamma_c(G)$. Hence, $\gamma_t(G) = \gamma_c(G)$.

Lemma 7. *Let G be a unicyclic graph with cycle C_m. If $5 \leq |X| \leq m - 2$, then $\gamma_t(G) \neq \gamma_c(G)$.*

Proof. As regards the longest path in $\langle X \rangle$, we can discuss it using the following cases.

Case 1 $t \geq 5$. Let $S = I(G) - \{v_{t-1}, v_t\}$. Since $\gamma_c(G) = I(G) - 2$, it follows that S is a γ_c-set of G. It is obvious that $S - \{v_1\}$ is a total dominating set of G. Hence $\gamma_t(G) < \gamma_c(G)$.

Case 2 $t = 4$. Let $S = I(G) - \{v_2, v_3\}$. Since $\gamma_c(G) = I(G) - 2$, it follows that S is a γ_c-set of G. Since $|X| \geq 5$, there exists at least a vertex $v_i \in X \cap S$, where $5 < i < m$. It is obvious that $S - \{v_i\}$ is a total dominating set of G. Hence $\gamma_t(G) < \gamma_c(G)$.

Case 3 $t = 3$. Let $S = I(G) - \{v_2, v_3\}$. Since $\gamma_c(G) = I(G) - 2$, it follows that S is a γ_c-set of G. Since $|X| \geq 5$, there exist at least two vertices $v_i \in X \cap S$, where $4 < i < m$. Suppose v_i is the first vertex of degree 2 in the path $v_m, v_{m-1}, \ldots, v_4$. It is obvious that $S - \{v_i\}$ is a total dominating set of G. Hence $\gamma_t(G) < \gamma_c(G)$.

Case 4 $t = 2$. Let $S = I(G) - \{v_1, v_2\}$. Since $\gamma_c(G) = I(G) - 2$, it follows that S is a γ_c-set of G. Since $|X| \geq 5$, there exist at least three vertices $v_i \in X \cap S$, where $3 < i < m$. Suppose v_i is the second vertex of degree 2 in the path $v_m, v_{m-1}, \ldots, v_3$. It is obvious that $S - \{v_i\}$ is a total dominating set of G. Hence $\gamma_t(G) < \gamma_c(G)$.

Case 5 $t = 1$. Let $S = I(G) - \{v_1\}$. Since $\gamma_c(G) = I(G) - 1$, it follows that S is a γ_c-set of G. Since $|X| \geq 5$, there exist at least four vertices $v_i \in X \cap S$, where $2 < i < m$. Suppose v_i is the second vertex of degree 2 in the path $v_m, v_{m-1}, \ldots, v_3$. It is obvious that $S - \{v_i\}$ is a total dominating set of G. Hence $\gamma_t(G) < \gamma_c(G)$.

With a similar proof to those of Theorem 1 and Lemma 7, the following two lemmas hold.

Lemma 8. *Let G be a unicyclic graph with cycle C_m. If $|X| = 0$, then $\gamma_t(G) = \gamma_c(G)$ if and only if $I(G) = S(G) \cup C(G)$.*

Lemma 9. *Let G be a unicyclic graph with cycle C_m and $|X| \leq m - 2$. If $t = 1$ and $|X| = 4$, then $\gamma_t(G) \neq \gamma_c(G)$.*

Lemma 10. *Let G be a unicyclic graph with cycle C_m and $|X| \leq m-2$. Suppose $t = 1$ and $1 \leq |X| \leq 3$. Let $G' = G - \{v_1\}$. Then $\gamma_t(G) = \gamma_c(G)$ if and only if $I(G') = S(G') \cup C(G')$.*

Proof. It is obvious that $\gamma_c(G) = \gamma_c(G') = |I(G)| - 1 = |I(G')|$.

Let $\gamma_t(G) = \gamma_c(G)$. If there exists a vertex $v \in I(G') - S(G') \cup C(G')$, then by Theorem 1, $\gamma_t(G') < \gamma_c(G')$. Let S be a minimum total dominating set of G' such that $S \cap L(G') = \emptyset$. Then $S \subseteq I(G')$. If $S \cap \{v_2, v_m\} \neq \emptyset$, then S is a total dominating set of G. Hence, $\gamma_t(G) < \gamma_c(G)$, which is a contradiction. Suppose $S \cap \{v_2, v_m\} = \emptyset$. Then $|S| \leq |I(G')| - 2$ and $S \cup \{v_m\}$ is a total dominating set of G. Hence, $\gamma_t(G) < \gamma_c(G)$, which is a contradiction. So, $I(G') = S(G') \cup C(G')$.

Conversely, if $I(G') = S(G') \cup C(G')$, then $\gamma_t(G') = \gamma_c(G')$. So, $I(G')$ is a minimum cardinality total dominating set of G'. If $\gamma_t(G) < \gamma_t(G')$, then there exists at least one vertex $v \in I(G')$ such that $I(G') - \{v\}$ is a total dominating set of G. Since $v = v_1$, it follows that $I(G') - \{v\}$ is also a total dominating set of G', which is a contradiction. Hence, $\gamma_t(G) = \gamma_t(G')$. So, $\gamma_t(G) = \gamma_c(G)$.

Lemma 11. *Let G be a unicyclic graph with cycle C_m and $|X| \leq m - 2$. Suppose $t = |X| = 2$. Let $G' = G - \{v_1\}$. Then $\gamma_t(G) = \gamma_c(G)$ if and only if $I(G') = S(G') \cup C(G') \cup \{v_m\}$.*

Proof. It is obvious that $\gamma_c(G) = \gamma_c(G') = |I(G)| - 2 = |I(G')|$.

Let $\gamma_t(G) = \gamma_c(G)$. If there exists a vertex $v \in I(G') - S(G') \cup C(G') \cup \{v_m\}$, then by Theorem 1 $\gamma_t(G') < \gamma_c(G')$, and there exists a total dominating set S of G' such that $S \subseteq I(G') - \{v\}$. If $v_m \in S$, then S is a total dominating set of G. Hence, $\gamma_t(G) < \gamma_c(G)$, which is a contradiction. If $v_m \notin S$, then $|S| \le |I(G')| - 2$ and $S \cup \{v_m\}$ is a total dominating set of G. Hence, $\gamma_t(G) < \gamma_c(G)$, which is a contradiction.

Conversely, if $v_m \in S(G') \cup C(G')$, similarly to Lemma 11, it follows that $\gamma_t(G) = \gamma_c(G)$. If $v_m \notin S(G') \cup C(G')$, then it is obvious that $I(G') - \{v_m\}$ is a minimum cardinality total dominating set of G'. That is, $\gamma_t(G') = \gamma_c(G') - 1$. Since $\gamma_t(G) = \gamma_t(G') + 1$, it follows that $\gamma_t(G) = \gamma_c(G)$.

Lemma 12. *Let G be a unicyclic graph with cycle C_m and $|X| \le m-2$. Suppose $t = 2$ and $3 \le |X| \le 4$. Let $G' = G - \{v_1, v_2\}$. Then $\gamma_t(G) = \gamma_c(G)$ if and only if $I(G') = S(G') \cup C(G')$.*

Proof. It is obvious that $\gamma_c(G) = \gamma_c(G') = |I(G)| - 2 = |I(G')|$.

Suppose $\gamma_t(G) = \gamma_c(G)$. Then any $v_i \in X \setminus \{v_1, v_2\}$ is adjacent to at least one vertex of $\{v_3, v_m\}$ and there exists v_j such that $N_{G'}(v_j) \setminus V(C_m) \subseteq L(G')$, where $j = 3, m$. Assume $v_{m-1} \in X$ is adjacent to v_m and $N_{G'}(v_m) \setminus V(C_m) \subseteq L(G')$. If there exists a vertex $v \in I(G') - S(G') \cup C(G')$, then by Theorem 1, $\gamma_t(G') < \gamma_c(G')$, and there exists a total dominating set S of G' such that $S \subseteq I(G') - \{v\}$.

If $|X| = 3$ and $m \ge 6$, then $v_m, v_{m-1} \in S$ and $(S \setminus \{v_{m-1}\}) \cup \{v_1\}$ is a total dominating set of G. Hence, $\gamma_t(G) < \gamma_c(G)$, which is a contradiction. If $|X| = 3$ and $m = 5$ or $|X| = 4$, then $v_m, v_3 \in S$ and S is a total dominating set of G. Hence, $\gamma_t(G) < \gamma_c(G)$, which is a contradiction.

Conversely, if $I(G') = S(G') \cup C(G')$, then $\gamma_t(G') = \gamma_c(G')$. Similarly to Lemma 11, it follows that $\gamma_t(G) = \gamma_c(G)$.

In a similar way to that above, we can prove the following lemma.

Lemma 13. *Let G be a unicyclic graph with cycle C_m and $|X| \le m - 2$. Suppose $t = |X| = 3, 4$. Let $G' = G - \{v_2\}$. Then $\gamma_t(G) = \gamma_c(G)$ if and only if $I(G') = S(G') \cup C(G')$.*

Let η denote the set of graphs such that each graph is obtained from C_6 by attaching at least a pendant vertex to v_1 and v_3.

Lemma 14. *Let G be a unicyclic graph with cycle C_m and $|X| \le m - 2$. Suppose $t = 3$ and $|X| = 4$. Then $\gamma_t(G) = \gamma_c(G)$ if and only if $G \in \eta$.*

Proof. Suppose $m \ge 7$. Each γ_c-set of G contains at least four vertices of $V(C_m) - \{v_1, v_2, v_3\}$. Without loss of generality, we can assume $v_{m-1} \in X$. Let $S = I(G) - \{v_2, v_3\}$. Since $\gamma_c(G) = I(G) - 2$, it follows that S is a γ_c-set of G. It is obvious that $S - \{v_{m-1}\}$ is a total dominating set of G. Hence $\gamma_t(G) < \gamma_c(G)$, which is a contradiction. Hence $m = 6$. It is obvious that $G \in \eta$.

Conversely, if $G \in \eta$, then it is obvious that $\gamma_t(G) = \gamma_c(G)$.

Theorem 2. *Let G be a unicyclic graph with cycle C_m and $|X| \le m - 2$. Then $\gamma_t(G) = \gamma_c(G)$ if and only if G is isomorphic to one graph of η, or one of the following conditions holds:*

(a) *Suppose $|X| = 0$. Then $I(G) = S(G) \cup C(G)$.*

(b) *Suppose $t = 1$ and $1 \le |X| \le 3$. Let $G' = G - \{v_1\}$. Then $I(G') = S(G') \cup C(G')$.*

(c) *Suppose $t = 2$ and $3 \le |X| \le 4$. Let $G' = G - \{v_1, v_2\}$. Then $I(G') = S(G') \cup C(G')$.*

(d) *Suppose $t = |X| > 1$. Let $G' = G - \{v_2\}$. If $|X| = 2$, then $I(G') = S(G') \cup C(G') \cup \{v_3\}$. If $3 \le |X| \le 4$, then $I(G') = S(G') \cup C(G')$.*

Acknowledgement

This work was supported by National Sciences Foundation of China (19871036).

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